Stochastic resonance and symmetry breaking in a one-dimensional system

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We derive and discuss properties of an exact solution for the average time for trapping of a Brownian particle driven by a random, asymmetric but unbiased, telegraph signal. The particle moves along a line segment terminated by either two traps or a trap and a reflecting point. Numerical results suggest that stochastic resonance, defined as a nonmonotonic behavior of the mean trapping time, is absent in the first case but present in the second. This generalizes a result obtained earlier by Doering and Gadoua [Phys. Rev. Lett. **69**, 2318 (1992)] and implies that symmetry breaking alone does not necessarily create stochastic resonance. [S1063-651X(97)13509-7]

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I. INTRODUCTION

Stochastic resonance (SR) is a widely investigated phenomenon with an increasing number of applications in physics, chemistry, and biology [1]. Although the terminology is not universally agreed on, SR generally refers to an enhancement, through an increase of noise amplitude, of some desirable characteristic of the output of a dynamical system subject to either a periodic or random field. Different manifestations of SR have been demonstrated in specific linear and nonlinear systems by analytic methods, and have also been shown to occur experimentally in biological [2] and physical [3] systems. However, there is still incomplete understanding of the conditions that may be necessary to produce SR in spite of the large variety of models of SR that have been analyzed. Here we define SR to be a nonmonotonic dependence of an output signal on one or more parameters that characterize an external driving force. A description of some systems producing this generalized form of SR is given in [3].

Our general identification of SR with nonmonotonic parametric variation allows for the occurrence of SR in some linear systems. A simple example of SR in a linear system is that of one-dimensional diffusion on a segment terminated by one or two traps with a periodic forcing term [4]. This system may be said to be characterized by two states, one in which the particle is untrapped and the second in which it is trapped. Both the first and second moments of the time to trapping for this system have been shown to exhibit SR [4].

In order to have SR in a system, one apparently needs a characteristic time or, equivalently, an external frequency associated with the driving force. When a one-dimensional diffusive system with one or two traps is also driven by an external periodic field, the mean first-passage time (MFPT) and the variance of the trapping time has been shown to vary nonmonotonically as a function of both the frequency [4] and amplitude [5] of the periodic force. However, the external force need not be periodic to define a characteristic frequency [3], since this can be defined as the reciprocal of the fluctuation rate. This raises the possibility of having SR in a system driven by two different types of noise, e.g., the combination of white and colored noise. The first study of properties of such systems was reported on by Balakrishnan, van den Broeck, and Hänggi [6] but the topic of SR in such systems was not raised. A generalization of this system was investigated by Doering and Gadoua [7], who considered the jumps in a linear double-well potential when the slope of the potential randomly fluctuates between two values at a rate γ. Somewhat surprisingly, SR in the form of a nonmonotonic dependence of the MFPT on γ occurred when one of the boundaries was absorbing and the other reflecting [7], but not when both boundary points were absorbing [8]. In the present paper we reanalyze the system studied in Ref. [6], showing that it, too, appears not to exhibit resonant behavior when the end points are both traps, but does exhibit SR when one endpoint is a trap and the second is a reflecting point.

Because of the large variety of resonant behavior, it is hardly surprising that globally necessary conditions for the occurrence of SR have not been identified. In particular, the role of nonsymmetric boundary conditions remains unclear, as is the type and degree of ''symmetry breaking'' required in the definition of the dynamical system in order for it to exhibit SR. An indication of some of the difficulties that arise in defining what is meant by symmetry breaking is found in a model whose properties were studied by Brey and Cassado-Pascual [9]. These investigators studied the properties of a random walk on a finite interval terminated by one or two traps. This model is somewhat similar to the model of Fletcher, Havlin, and Weiss [4], but instead of having the system driven by an oscillating field, the authors allow the system to evolve on a line in which at any time each site has one of two transition rates, these being allowed to change at random times. In the system of Brey and Cassado-Pascual, SR is found to occur when one of the endpoints is reflecting and the other absorbing, but not when both are absorbing as in the paper by Doering and Gadoua [7]. In this paper we analyze a model in which the deterministic telegraph signal is replaced by a random one. One result of the present paper is the suggestion that SR does not occur with two absorbing points even with an asymmetric but unbiased telegraph signal, but that SR does occur when one of the boundaries is absorbing and the second reflecting as in Ref. [7].

II. THE SYSTEM DEFINED

We consider a stochastic one-dimensional motion $x(t)$ whose evolution is determined by a sum of two types of noise, one of which is Gaussian white noise and the second an asymmetric random telegraph signal. The state variable $x(t)$ in the overdamped case therefore satisfies $\dot{x}=\eta(t)+f(t)$ where $f(t)$ is zero-mean uncorrelated white noise whose second-order moment properties are $\langle f(t)f(t')\rangle=2D\delta(t-t')$, in which D is a diffusion constant. The second component of the noise, $\eta(t)$, is chosen to be an asymmetric randomly switching telegraph signal that can assume either of the values v_+ or $-v_-$ where both velocities are positive.

A simple model whose properties can be determined is one in which switching between the two states determined by the telegraph signal occurs at random spaced times whose properties are described by negative exponential probability densities $\vartheta_i(t)=T_i^{-1}\exp(-t/T_i)$, $i=+$ or $-$ so that T_i is the average time spent by the particle in a single sojourn in state i. The random telegraph signal $\eta(t)$ can also be defined in terms of its first two moments, viz.,

$$\langle\eta(t)\rangle=\frac{v_+T_+-v_-T_-}{T_++T_-},$$

$$\langle\eta(t)\eta(t')\rangle=\langle\eta(t)\rangle\langle\eta(t')\rangle+\frac{T_+T_-(v_+-v_-)^2}{(T_++T_-)^2}e^{-|t-t'|/T}, \tag{1}$$

in which $T^{-1}\equiv T_+^{-1}+T_-^{-1}$ is a relaxation rate for the correlation function.

We calculate the MFPT, $\tau(x_0)$, for the time to trapping of a particle, initially at x_0, diffusing on a line segment $(0, L)$ terminated by either two traps or a single trap and a reflecting point.

III. ANALYSIS

A. Basic equations

Let $\tau_+(x_0)$ be the MFPT for a particle which is initially located at x_0 and whose initial velocity is v_+, with a similar definition for $\tau_-(x_0)$. For convenience, we assume that the two states are initially equally likely. The coupled set of equations for $\tau_+(x_0)$ and $\tau_-(x_0)$ is

$$D\frac{d^2\tau_+}{dx_0^2}+v_+\frac{d\tau_+}{dx_0}+\frac{\tau_-}{T_-}-\frac{\tau_+}{T_+}=-\frac{1}{2}, \tag{2}$$

$$D\frac{d^2\tau_-}{dx_0^2}-v_-\frac{d\tau_-}{dx_0}+\frac{\tau_+}{T_+}-\frac{\tau_-}{T_-}=-\frac{1}{2}, \tag{3}$$

where the initial equally likely occurrence of both states requires the right-hand sides of these equations to be equal.

For purposes of analysis, it is convenient to replace Eqs. (2) and (3) by an equivalent set for the unconditional MFPT $\tau(x_0)$ and an auxiliary function $\Gamma(x_0)$ having the dimensions of length. These are defined by

$$\begin{aligned}\tau(x_0)&=\tau_+(x_0)+\tau_-(x_0),\\ \Gamma(x_0)&=v_+\tau_+(x_0)-v_-\tau_-(x_0).\end{aligned} \tag{4}$$

By solving this set of equations for $\tau_+(x_0)$ and $\tau_-(x_0)$ in terms of $\tau(x_0)$ and $\Gamma(x_0)$ and adding and subtracting Eqs. (2) and (3), we can replace that set of equations by

$$D\tau''+\Gamma'=-1, \tag{5}$$

$$D\Gamma''+\frac{D}{2}(v_--v_+)\tau''+\left(\frac{v_+-v_-}{2}\right)\Gamma'+v_+v_-\tau' -\left(\frac{1}{T_+}+\frac{1}{T_-}\right)\Gamma+\left(\frac{v_+}{T_-}-\frac{v_-}{T_+}\right)\tau=0. \tag{6}$$

When the two endpoints are traps and $D\neq 0$, Eqs. (5) and (6) are to be solved subject to the boundary conditions $\tau(0)=\tau(L)=\Gamma(0)=\Gamma(L)=0$. When $x_0=0$ is a trapping point and $x_0=L$ a reflecting one the spatial derivatives of τ and Γ vanish at $x_0=L$. The resulting set of equations are just those originally studied by Doering and Gadoua [7].

Equations (5) and (6) can be combined into a single fourth-order differential equation for $\tau(x_0)$. We consider only the case in which the time-averaged mean bias is equal to 0 or $\langle\eta(t)\rangle=0$. The resulting equation is

$$D^2\tau''''+D(v_+-v_-)\tau'''-\left[v_+v_-+\frac{D}{T}\right]\tau''=\frac{1}{T}. \tag{7}$$

This is a linear differential equation solvable by elementary methods. The solutions themselves are rather formidable and are therefore relegated to the Appendix.

B. Summary of numerical results

The exact expressions given in the Appendix are so complicated that we were unable to rigorously prove the nonexistence of SR in the two-trap case. However, properties of $\tau(x_0)$ considered as functions of v_+, v_+/v_-, and T were investigated over a wide range of these parameters, finding no indication of SR in any of them when the two endpoints of the line are traps. If this is generally true, it follows that the existence of two states alone is insufficient to produce SR just as was found by Doering and Gadoua [7] and Brey and Casado-Pascual [9].

Another type of symmetry breaking is introduced by assuming that the boundary conditions at the two endpoints differ. Here, again in agreement with the investigations in Refs. [7] and [9], one indeed finds SR. Figure 1 shows curves of $\tau(1/2)$ as a function of the switching rate T for different amplitudes of the velocities, which are taken to be equal in our calculations $(v_+=v_-)$. Because the SR is due to the coherence of motion as in Ref. [4], it becomes more pronounced as either the noise or velocity amplitudes increase. In Fig. 2 we again plot $\tau(1/2)$ as a function of T, this time allowing the v's to differ so that yet another source of symmetry breaking is introduced. Because of the asymmetry in the endpoints, the curves corresponding to $v_+/v_-=2$ differ from those when $v_+/v_-=1/2$. That is to say, asymmetry in the velocities can serve to either increase or lessen the

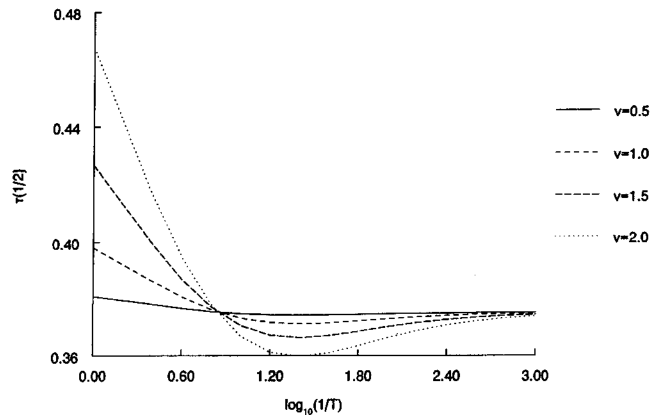

FIG. 1. Curves of $\tau(1/2)$ plotted as a function of $\log_{10}(1/T)$ for different values of $v=v_+=v_-$. The numerical results indicate that in each case SR will exist, but larger velocities lead to more noticeable minima in the curves.

MFPT, but does not change the fact of whether SR does or does not occur.

Figures 1 and 2 have demonstrated nonmonotonic behavior in $\tau(1/2)$ as a function of T for different conditions imposed on the velocities. Nonmonotonic behavior also exists as a function either as the amplitude of the velocity changes in the case of a symmetric telegraph signal [Fig. 3(a)] or as the degree of asymmetry between the two velocities changes as illustrated by the curves in Fig. 3(b). Finally, Fig. 3(c) shows the SR obtained for $\tau(1/2)$ as a function of v_- for different values of v_+/T_- [10–13].

IV. CONCLUDING REMARKS

What we have shown is that for the system studied here, symmetry-breaking is required to produce SR but the specific parameter in which symmetry breaking occurs does matter. Here we have considered various aspects of coherent SR, in terms of the MFPT for the survival of a particle. Similar problems arise in the more commonly used definition of SR in problems in terms of motion in double-well systems subject to the combination of periodic and random signals

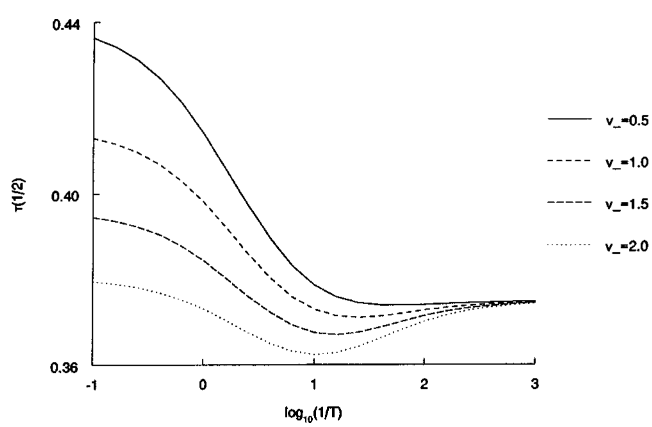

FIG. 2. Curves of $\tau(1/2)$ plotted as a function of $\log_{10}(1/T)$ for $v_+=1$ for different values of v_-. SR exists in each of the cases shown.

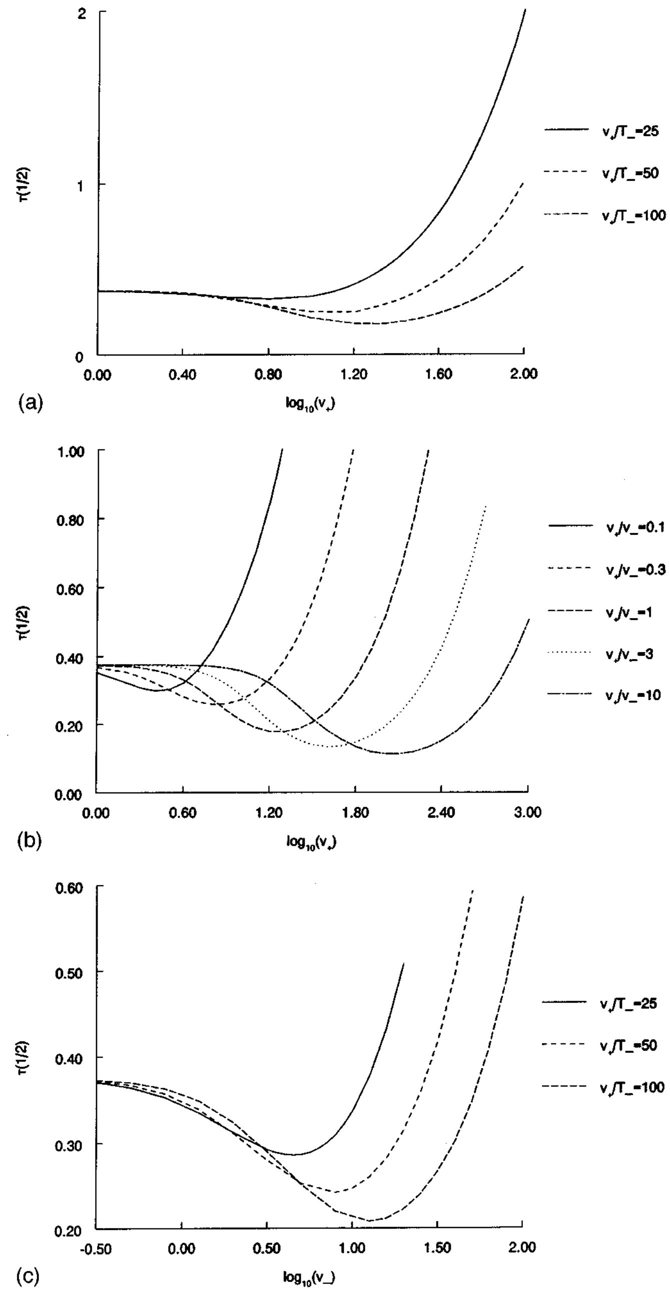

FIG. 3. The existence of SR for $\tau(1/2)$ considered as functions of the velocities. (a) $\tau(1/2)$ plotted as a function of $\log_{10}(v_+)$ for the symmetric telegraph signal showing SR in each of the indicated three cases. (b) SR demonstrated for the asymmetric telegraph signal with differing ratios of the velocities. (c) $\tau(1/2)$ plotted as a function of $\log_{10}(v_-)$ again demonstrating the existence of resonant behavior.

[14]. These authors concluded that SR does not occur in the presence of a periodic perturbation. Another example making the same point is that of motion in a double-well potential subject to the combination of multiplicative noise and a multiplicative periodic force as studied by Dykman *et al.* [15]. These authors found that SR could be produced by fixing the two wells to have different depths. As a final remark, it might also be interesting to investigate whether SR appears in a system in which one of the boundaries is partially, rather than fully, reflecting.

APPENDIX: EXPRESSIONS FOR $\tau(x_0)$ FOR THE TWO TYPES OF BOUNDARY CONDITIONS

1. Two traps

All of the equations here are expressed in terms of dimensionless variables, in which time is measured in terms of L^2/D and lengths are given in terms of L. This scaling is equivalent to setting $D=L=1$. The most general solution of Eq. (7) subject to trapping boundary conditions at both ends of the line will be expressed in terms of the functions

$$R_{\pm}(z)=\frac{\exp(\lambda_{\pm}z)-1}{\exp(\lambda_{\pm})-1},\quad 0\leqslant z\leqslant 1,$$
$$B_{\pm}=\frac{\lambda_{\pm}}{T}, \tag{A1}$$

where $\lambda_{\pm}=\alpha\pm\beta$, the parameters α and β being

$$\alpha=\frac{(v_{-}-v_{+})}{2D}=\frac{v_{+}}{2D}\left(\frac{T_{+}}{T_{-}}-1\right), \tag{A2}$$
$$\beta=\frac{1}{2D}\left[v_{+}^{2}\left(1+\frac{T_{+}}{T_{-}}\right)^{2}+\frac{4D}{T}\right]^{1/2}.$$

Equation (A1) is written in terms of dimensionless variables. The functions $R_{\pm}(z)$ clearly satisfy $R_{\pm}(0)=0$ and $R_{\pm}(1)=1$. The full expression for $\tau(x_0)$ is

$$\tau(x_0)=\frac{\gamma x_0(1-x_0)}{2T}-\frac{\gamma v_{+}v_{-}}{2\beta}\left[R_{+}(x_0)-R_{-}(x_0)\right]+C\left[x_0+\frac{\lambda_{-}R_{+}(x_0)-\lambda_{+}R_{-}(x_0)}{2\beta}\right], \tag{A3}$$

where $\gamma=(v_{+}v_{-}+1/T)^{-1}$ and C is

$$C=\gamma\,\frac{\beta(v_{+}-v_{-})\left(\frac{1}{T}-v_{+}v_{-}\right)(\cosh\beta-\cosh\alpha)-\frac{v_{+}v_{-}}{T}(\alpha\,\sinh\beta-\beta\,\sinh\alpha)}{2\beta v_{+}v_{-}(\cosh\beta-\cosh\alpha)+(\gamma T)^{-1}\,\sinh\beta}. \tag{A4}$$

It is readily verified that $\tau(0)=\tau(1)=0$.

2. One trap and a reflecting point

Here we set the trap at $z=0$ and the reflecting point at $z=1$. Define the constants

$$Q=v_{+}v_{-}T,\quad S=Q+\alpha T(Q-1),\quad \gamma=T/(1+Q), \tag{A5}$$
$$B=\lambda_{\pm}^{2}\,\exp(\lambda_{\pm}),\quad A_{\pm}=\lambda_{\pm}+QB_{\pm}/\lambda_{\pm},$$
$$\Omega_{\pm}=SB_{\pm}+QA_{\pm}$$

and the functions $U_{\pm}(x_0)=\exp(\lambda_{\pm}x_0)-x_0B_{\pm}-1$. The formula for $\tau(x_0)$ can be written in terms of these as

$$\tau(x_0)=\gamma\,\frac{x_0(2-x_0)}{2T}+\gamma\,\frac{\Omega_{-}U_{+}(x_0)-\Omega_{+}U_{-}(x_0)}{T(A_{+}B_{-}-A_{-}B_{+})}, \tag{A6}$$

generalizing a result given by Doering and Gadoua [7].

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